

Merger Rates of Dark-Matter Haloes

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ABSTRACT

We derive analytic merger rates for dark-matter haloes within the framework of the Extended Press-Schechter (EPS) formalism. These rates become self-consistent within EPS once we realize that the typical merger in the limit of a small time-step involves more than two progenitors, contrary to the assumption of binary mergers adopted in earlier studies. We present a general method for computing merger rates that span the range of solutions permitted by the EPS conditional mass function, and focus on a specific solution that attempts to match the merger rates in N -body simulations. The corrected EPS merger rates are more accurate than the earlier estimates of Lacey & Cole, by $\sim 20\%$ for major mergers and by up to a factor of ~ 3 for minor mergers of mass ratio $1 : 10^4$. Based on the revised merger rates, we provide a new algorithm for constructing Monte-Carlo EPS merger trees, that could be useful in Semi-Analytic Modeling. We provide analytic expressions and plot numerical results for several quantities that are very useful in studies of galaxy formation. This includes (a) the rate of mergers of a given mass ratio per given final halo, (b) the fraction of mass added by mergers to a halo, and (c) the rate of mergers per given main progenitor. The creation and destruction rates of haloes serve for a self-consistency check. Our method for computing merger rates can be applied to conditional mass functions beyond EPS, such as those obtained by the ellipsoidal collapse model or extracted from N -body simulations.

Key words: cosmology: theory — dark matter — galaxies: haloes — galaxies: formation — gravitation

1 INTRODUCTION

The hierarchical clustering of dark matter is the key process in establishing the observed structure in the universe. Galaxies form inside the potential wells induced by the dark-matter distribution. The building blocks of this hierarchy are virialized collapsed gravitating systems in pressure equilibrium — the dark-matter haloes — characterized by their growth history, structure, and clustering. Although dark-matter dynamics is governed solely by the gravitational force, we are still far from a good quantitative understanding of its various features.

The Press-Schechter (PS) formalism (Press & Schechter 1974) has been very useful in modeling the abundance of dark-matter haloes as a function of mass and time. It has been further developed by Bond et al. (1991) and Lacey & Cole (1993, hereafter LC93) to the Extended Press-Schechter (EPS) formalism, which provides at any time the mass function of progenitors of a halo of a given current mass. EPS has been a basic tool for understanding the growth history of haloes, and it has been shown to grasp many of the key features of the buildup of haloes

in cosmological N -body simulations (e.g. Lacey & Cole 1994; Cole et al. 2008; Neistein & Dekel 2008). While EPS has been used extensively for the last two decades, it still involves central open issues. One is the construction of self-consistent Monte-Carlo merger trees for Semi-Analytic Models of galaxy formation. The other is how to compute halo merger rates that will be consistent with the EPS mass function.

While drawing the basic lines of the EPS theory, LC93 worked out a formula for the merger rates of haloes. This formula has been popular in many applications, although it involves a problem. LC93 themselves noticed that their merger-rate formula has a problematic intrinsic asymmetry between progenitors of mass M and $M_0 - M$ (where M_0 is the descendant halo mass). Sheth & Pitman (1997) realized that the LC93 assumption of binary mergers is not accurate when the power spectrum differs from a white noise (their discussion near eq. 27). Benson et al. (2005) interpreted this as an intrinsic inconsistency within the EPS formalism. We show below that the typical mergers have multiple progenitors, more than two, even in the limit of a small time step. Adopting this correct limit, we obtain accurate EPS merger rates, which improve the LC93 estimates and are fully consistent with the EPS conditional mass function. The error

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in the LC93 formula makes a significant difference for the number of merger events and for the fraction of halo mass added by mergers.

Random realizations of merger trees that follow the EPS conditional mass function are widely used as the back-bone of semi-analytic modeling of galaxy formation. Several different methods for constructing such trees have been proposed (Cole 1991; Kauffmann & White 1993; Sheth & Lemson 1999; Somerville & Kolatt 1999; Cole et al. 2000; Hiotelis & Popolo 2006). In most cases, these algorithms fail to recover the EPS mass function. It seems that the algorithm of Kauffmann & White (1993) is the only one that is fully consistent with EPS. Sheth & Lemson (1999) described an alternative that is also accurate, but it has not been developed into a detailed solution. One can indeed show that the EPS formalism permits many different types of merger trees that recover the EPS progenitor mass function. We provide below a new algorithm for constructing EPS merger trees based on our formula for merger rates. This algorithm does reproduce the EPS progenitor mass function, and it is chosen among the different solutions to be a good match to the merger trees extracted from cosmological N -body simulations.

Empirical algorithms for generating merger trees that resemble the trees in cosmological N -body simulations have been proposed by Parkinson et al. (2008); Neistein & Dekel (2008). These merger trees are for most parts better approximations to the N -body results than any EPS-based tree. However, a correct EPS model has several useful benefits. For example, it allows very high mass resolution at low cost, it can be easily applied within any desired cosmological model, and it is self-consistent with the Press-Schechter halo abundance. On the other hand, the empirical algorithms mentioned above should always be verified and re-calibrated when used in a different cosmology or when applied at a different resolution. Analytic models in the spirit of EPS can serve us in understanding several open issues concerning the way haloes are identified in N -body simulations. For example, it has been noticed (Neistein & Dekel 2008, hereafter ND08) that some of the non-Markov features in N -body merger trees may arise from the way haloes are defined. Indeed, the halo definition has become an open issue with the finding that the range of virial equilibrium in small haloes can extend well beyond the traditional “virial radius” that is based on spherical collapse (Cuesta et al. 2007; Ludlow et al. 2008). As part of this work we provide a general method for generating merger trees that follow any given conditional mass function. This mass function could be either based on spherical collapse (i.e., EPS), or arise from ellipsoidal collapse (Sheth & Tormen 2002), or extracted from N -body simulations.

This paper is organized as follows. In §2 we present nomenclature, describe the limit of small time-steps, and prove the theorem concerning multiple progenitors. In §3 we address different solutions for the EPS halo merger rates, and choose the solution that fits well the N -body results. In §4 we work out useful results for merger rates from our EPS formalism, and present them in practical formulae and in figures. In §5 we address the creation and destruction rates of haloes. In §6 we describe a Monte-Carlo algorithm for constructing EPS merger trees based on our adopted solution. In §7 we summarize our results and discuss them.

2 GENERAL ANALYSIS

2.1 Definitions: PS and EPS

In the EPS formalism, the natural dimensionless time variable is $\omega(z) = \delta_c(z)/D(z)$, where $D(z)$ is the cosmological linear growth rate of density fluctuations as a function of redshift z and $\delta_c \simeq 1.69$. The natural mass variable is $S(M) = \sigma^2(M)$, the variance of the initial density fluctuation field, linearly extrapolated to $z = 0$, and smoothed using a window function that corresponds to a mass M . The reader is referred to ND08 for our specific way for computing these quantities. The cosmological model used here is defined by $(\Omega_\Lambda, \Omega_m, h, \sigma_8) = (0.75, 0.25, 0.73, 0.9)$, with the power spectrum specified in ND08. This model was adopted to enable comparison with results extracted from the Millennium cosmological simulation (Springel et al. 2005).

According to the EPS formalism (Bond et al. 1991, LC93), the average number of progenitors in the mass interval $[M, M + dM]$, which will merge into a descendant halo M_0 after a time-step $\Delta\omega$, is given by

$$\frac{dN}{dM}(M|M_0, \Delta\omega) dM = \frac{M_0}{M} \frac{1}{\sqrt{2\pi}} \frac{\Delta\omega}{(\Delta S)^{3/2}} \exp\left[-\frac{(\Delta\omega)^2}{2\Delta S}\right] \left|\frac{dS}{dM}\right| dM, \quad (1)$$

where $\Delta S = S(M) - S(M_0)$. We term the most massive progenitor in this time-step by M_1 , the second most massive by M_2 , and so on. The probability that M is the mass of the i -th progenitor is termed $P_i = P_i(M|M_0, \Delta\omega)$. Consequently, the sum of all the P_i ’s equals dN/dM :

$$\begin{aligned} P_{\text{tot}}(M|M_0, \Delta\omega) &\equiv \frac{dN}{dM}(M|M_0, \Delta\omega) \\ &= \sum_i P_i(M|M_0, \Delta\omega). \end{aligned} \quad (2)$$

For brevity, we may sometimes omit the explicit dependence of P_{tot} and P_i on M_0 and $\Delta\omega$.

It is often useful to define a minimum halo mass, M_{min} . Haloes with smaller masses are considered to be part of a smooth accretion component, encompassing a total mass M_{acc} .

We also need the total number density of haloes per unit mass per comoving volume, which is given by the Press-Schechter mass function:

$$\phi(M, z) = \frac{1}{\sqrt{2\pi}} \frac{\rho_0}{M} \frac{\omega}{S^{3/2}} \exp\left[-\frac{\omega^2}{2S}\right] \left|\frac{dS}{dM}\right|, \quad (3)$$

where ρ_0 is the present mean mass density of the universe.

2.2 Number of Progenitors in a Small Time-Step

Throughout this work, we appeal to the limit of a small time-step, $\Delta\omega \rightarrow 0$, relevant for the derivative with respect to “time”, $d/d\omega$. For given M_0 and M_{min} , the limit of a small time-step is defined here as $\Delta\omega \ll S(M_0 - M_{\text{min}}) - S(M_0)$. In this limit, and when $M \leq M_0 - M_{\text{min}}$, the probability P_{tot} can be written as

$$P_{\text{tot}}(M|M_0, \Delta\omega \rightarrow 0) = \frac{1}{\sqrt{2\pi}} \frac{M_0}{M} \frac{\Delta\omega}{(\Delta S)^{3/2}} \left|\frac{dS}{dM}\right|, \quad (4)$$

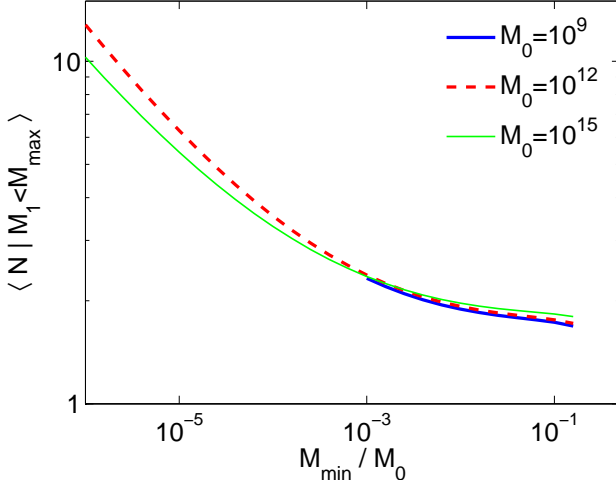


Figure 1. The average number of progenitors given that the main progenitor mass is less than $M_{\max} = M_0 - M_{\min}$, as a function of M_{\min}/M_0 . The three different curves are for different values of M_0 as indicated (with units of $h^{-1}M_\odot$). Each of the curves is plotted only for $M_{\min} > 10^6 h^{-1}M_\odot$. The computation is done in the limit of a small time-step, $\Delta\omega \rightarrow 0$. Evidently, if the minimum mass is less than $\sim 10^{-3}M_0$, the number of progenitors is larger than two. This implies that the concept of binary mergers is highly inaccurate for low values of M_{\min}/M_0 . This conclusion is valid independently of the value of M_0 .

after the exponent in eq. 1 has been set to unity. Consequently, the “time” derivative of P_{tot} is simply

$$\frac{dP_{\text{tot}}(M|M_0)}{d\omega} = \frac{1}{\sqrt{2\pi}} \frac{M_0}{M} \frac{1}{(\Delta S)^{3/2}} \left| \frac{dS}{dM} \right|. \quad (5)$$

We occasionally write $d/d\omega$ when it should formally be $d/d\Delta\omega$, as both derivatives are the same¹. The above equations are valid only for $M \leq M_0 - M_{\min}$; otherwise ΔS may also become infinitely small, such that $(\Delta\omega)^2/\Delta S$ does not vanish, and the exponent in eq. (1) does not converge to unity.

We now prove the theorem of multiple progenitors, claiming that according to EPS, the typical merger involves multiple progenitors rather than a binary merger even in the limit of a small time-step. **Theorem: Given the EPS progenitor mass function of eq. (1), with the CDM power-spectrum, in the range $M_{\min} \ll M_0$ and in the limit $\Delta\omega \rightarrow 0$, the average number of progenitors per merger event is greater than two.**

We first notice that the constraint of mass conservation, that the total mass in progenitors cannot exceed M_0 , implies that events with $M_1 > M_{\max}$, where $M_{\max} \equiv M_0 - M_{\min}$, cannot have any other progenitor with $M_i > M_{\min}$. Therefore, merger events between two or more progenitors above M_{\min} are limited to the cases where $M_1 < M_{\max}$.

Let N be the number of progenitors with mass in the range $[M_{\min}, M_{\max}]$. We first show that $\langle N | M_1 < M_{\max} \rangle > 2$. If $P(M_1 < M)$ is the probability that $M_1 < M$, then $\langle N \rangle = P(M_1 < M_{\max}) \times \langle N | M_1 < M_{\max} \rangle$, because the

contribution of the other events is zero progenitors². We thus obtain

$$\begin{aligned} \langle N | M_1 < M_{\max} \rangle &= \frac{\langle N \rangle}{P(M_1 < M_{\max})} \\ &= \frac{\int_{M_{\min}}^{M_{\max}} P_{\text{tot}}(M) dM}{1 - \int_{M_{\max}}^{M_0} P_1(M) dM}. \end{aligned} \quad (6)$$

When we calculate the integral in the denominator, we note that P_1 can be replaced by P_{tot} near M_0 (see the discussion preceding eq. 9 below). As $\Delta\omega \rightarrow 0$, eq. (4) implies that $\langle N \rangle$ vanishes in proportion to $\Delta\omega$, but $P(M_1 < M_{\max})$ also vanishes³, making the ratio converge to a finite value.

Figure 1 shows the average number of progenitors given that the main-progenitor mass is smaller than M_{\max} , $\langle N | M_1 < M_{\max} \rangle$. This is in the limit of a small time-step and for different values of M_0 . We see that this average is greater than two for any $M_{\min} < 10^{-3}M_0$. It increases with decreasing M_{\min} to a value of ~ 10 for $M_{\min} = 10^{-6}M_0$. This proves that $\langle N | M_1 < M_{\max} \rangle > 2$.

Since each of the events with $M_1 < M_{\max}$ that are not mergers contributes to the conditional average of N a value ≤ 1 , the merger events, which are a subset of the $M_1 < M_{\max}$ events, must have on average even more progenitors than computed in eq. (6) and shown in Fig. 1. We conclude that the assumption of binary mergers is invalid in EPS, even for $\Delta\omega \rightarrow 0$, once $M_{\min} < 10^{-3}M_0$. This proves the theorem.

If M_0 is not that much larger than M_{\min} , the range $M_{\min} \geq 10^{-2}M_0$ in Fig. 1, we obtain $\langle N | M_1 < M_{\max} \rangle \lesssim 2$. This implies that the average mass of the two progenitors does not sum up to M_0 , namely the accretion component M_{acc} contains a non-negligible fraction of the mass.

Since the theorem of multiple progenitors has interesting implications on the formation of structure, it would be worthwhile to consider analytically the average number of progenitors in the idealized case where the power spectrum is a pure power law, $S \propto M^{-\alpha}$. Solving for $\langle N | M_1 < M_{\max} \rangle$, we find that it is bigger than 2 for any $0 < \alpha < 1$ (once M_{\min} is small enough). For $\alpha = 1$, the case of Poisson white noise, one can show that $\langle N | M_1 < M_{\max} \rangle \rightarrow 2$ when $M_{\min}/M_0 \rightarrow 0$, in agreement with the coagulation approach discussed by Epstein (1983) and Sheth & Pitman (1997). For $\alpha > 1$, the average number of progenitors never exceeds 2. We learn that the average number of progenitors per merger event depends on the shape of the power spectrum. In particular, for power spectra that are relevant on galactic scales, $\alpha < 1$, the average number of progenitors per merger are more than two.

We note that the existence of multiple mergers in

² Strictly speaking, we should write $\langle N \rangle = P(M_{\min} < M_1 < M_{\max}) \times \langle N | M_{\min} < M_1 < M_{\max} \rangle$. Using the fact that $P(M_{\min} < M_1 < M_{\max}) \leq P(M_1 < M_{\max})$ it is evident that all the results proved here using $P(M_1 < M_{\max})$ are lower limits on the accurate $\langle N | M_{\min} < M_1 < M_{\max} \rangle$.

³ When computing the denominator one should use eq. 1, and not the approximation of eq. 4. In the case where $M_{\min} \ll M_0$ we can approximate $M_0/M \sim 1$ and the denominator is just $\text{Erf}[\Delta\omega/\sqrt{2\Delta S_m}] \sim \sqrt{2/\pi} \Delta\omega/\sqrt{\Delta S_m}$, where $\Delta S_m = S(M_{\max}) - S(M_0)$ and $\Delta\omega \rightarrow 0$. When M_{\min} is not small enough, the integral in the denominator can be computed only numerically.

¹ We assume that $\Delta\omega = \omega - \omega_0$ and the derivative $d/d\omega$ is computed at a fixed ω_0 .

the limit of small time-steps is already mentioned in Sheth & Pitman (1997). Sheth & Lemson (1999) added that for a general power spectrum, one can group the progenitors into sub-groups that merge like the progenitors of the Poisson-power-spectrum case.

2.3 Merger rates

One way to define a merger rate is as the probability for the i -th most massive progenitor to merge into the main progenitor within a time-step $\Delta\omega$. This is the joint probability for the two progenitor masses M_1 and M_i , which we denote $P_{1,i}(M_1, M_i|M_0, \Delta\omega)$. Note that P_i and P_j can both have non-vanishing values at the same mass, so the probability for *any* progenitor with mass M_s to merge with M_1 is the sum

$$P_{1,s}(M_1, M_s|M_0, \Delta\omega) \equiv \sum_i P_{1,i}(M_1, M_s|M_0, \Delta\omega). \quad (7)$$

We learned in §2.2 that there are typically several progenitors in each merger event even in the limit of a small time-step. To complicate matters even further, we note that $P_{1,i+1}$ is not necessarily smaller than $P_{1,i}$. Still, for the purpose of estimating merger rates, we wish to approximate this multi-progenitor merging process as an instantaneous sequence of binary mergers. There is clearly no unique way to do that. We adopt here the assumption that each of the secondary progenitors ($M_i, i > 1$) merges with a halo of mass M_1 , and ignore mergers among the secondary haloes themselves. The validity of this assumption can be tested in N -body simulations⁴. This assumption makes sense when M_1 is much more massive than the other progenitors. However, in a case where $M_1 \sim M_2 \gg M_3$, one might consider M_3 merging with a halo of mass $M_1 + M_2$ instead. We assume that this uncertainty in interpreting the multiple merger events does not translate to a significant error in our estimated average merger rate, but the actual estimate of this uncertainty is beyond the scope of the present paper.

For any progenitor M_i , we define $P_{i|1}(M_i|M_1, M_0, \Delta\omega)$ to be the conditional probability to have M_i *given* that the main-progenitor mass is M_1 . Then

$$P_{1,i}(M_1, M_i|M_0, \Delta\omega) = P_{i|1}(M_i|M_1, M_0, \Delta\omega) \cdot P_1(M_1|M_0, \Delta\omega). \quad (8)$$

Our approach for finding a solution for $P_{1,i}$ starts with a solution for P_1 , followed by a solution for $P_{i|1}$. This is because P_1 is determined robustly by EPS, with only a small, controllable uncertainty over a limited mass range.

The shape of P_1 , the small freedom in it within EPS, and its effect on the average mass history of the main progenitor has been studied in Neistein et al. (2006) and can be summarized as follows: In the range $M_1 \geq M_0/2$, P_1 is identical to the known P_{tot} , because any progenitor in this mass range is by definition the main progenitor. For M_1 slightly below

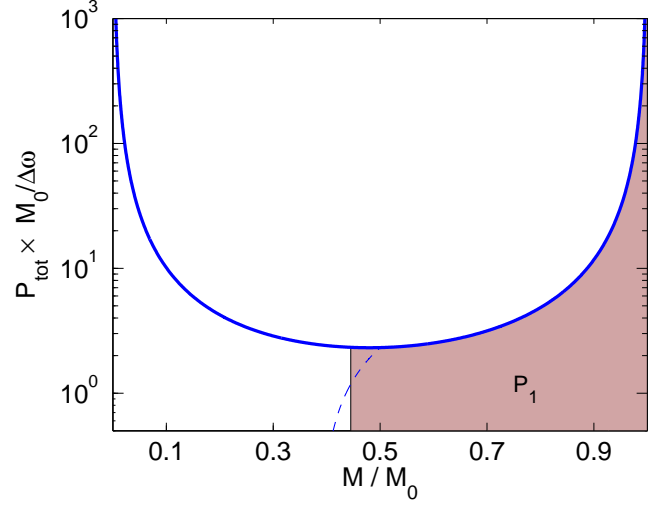


Figure 2. Two possible solutions for the probability distribution of the main progenitor, P_1 . The thick blue curve corresponds to P_{tot} , properly normalized as indicated with $M_0 = 10^{12} h^{-1} M_\odot$ and $\Delta\omega = 10^{-6}$. The solutions for P_1 differ only in the small tail at $M \lesssim M_0/2$. The shaded area marks the range over which the integral of P_{tot} equals unity; it ends at $x_1 = M/M_0 \sim 0.44$. This is also our default definition of P_1 , termed *sharp tail*. The dashed curve marks another possible tail, also corresponding to an integral of unity, which is linear in M , and thus termed *linear tail*. Note that we plot $P_{\text{tot}} \times M_0/\Delta\omega$ as this curve is the same for all small $\Delta\omega$, in accord with eq. 4.

$M_0/2$, there is a “tail” of non-vanishing probability, which could go to zero in many different ways. This is illustrated in Fig. 2, which shows two of the many possible solutions for this tail. Our default option is with a ‘sharp tail’,

$$P_1(M_1|M_0, \Delta\omega) = \begin{cases} P_{\text{tot}} & \text{if } M_1 > x_1 M_0 \\ 0 & \text{if } M_1 \leq x_1 M_0 \end{cases}. \quad (9)$$

The value of x_1 is set by the requirement that the integral over P_1 equals unity. It is $\simeq 0.44$ for the cosmology and for the halo masses used here⁵. We note that the average mass history of the main-progenitor using this P_1 can be computed by the analytical formula of Neistein et al. (2006). Fig. 2 also shows an alternative solution where the P_1 tail is linear in M . The freedom in the tail of P_1 corresponds to an uncertainty of less than 8% in the average relative growth rate of the main progenitor, $(dM_1/d\omega)/M_1$.

Assuming a specific solution for P_1 , the constraints for having a correct $P_{i|1}$ are as follows:

$$P_i(M_i|M_0) = \int P_{i|1}(M_i|M_1, M_0) P_1(M_1|M_0) dM_1 \quad (10)$$

$$P_{\text{tot}}(M) = \sum_i P_i(M) \quad (11)$$

$$P(M_1, M_2, \dots) = 0 \quad \text{if} \quad \sum_i M_i > M_0 \quad (12)$$

⁴ In §7, we discuss the possible relation between a multiple merger event in EPS and a correlated sequence of binary mergers in an N -body simulation. If the time between mergers in the N -body sequence is shorter than the time it takes the remnant halo to settle into its new potential, than our assumption might be reasonable.

⁵ Using $u = \log_{10}(M_0) - 12$ where M_0 is in units of $h^{-1} M_\odot$ we can approximate $x_1 = 7.118 \times 10^{-5} u^3 + 6.225 \times 10^{-4} u^2 + 0.0035 u + 0.444$ with an accuracy that is better than 0.05%.

The last condition is assuring mass conservation, where the total mass of all progenitors cannot be larger than M_0 .

For certain purposes, it will be helpful to define $P_{s|1}$ as the sum over all $P_{i|1}$. The constraint for $P_{s|1}$ is simply

$$P_{\text{tot}}(M) - P_1(M) = \int P_{s|1}(M|M_1)P_1(M_1)dM_1. \quad (13)$$

Here mass conservation cannot be formulated as an explicit condition on $P_{s|1}$ because it does not contain information concerning the mutual distribution of multiple progenitors.

While P_1 is robustly determined in EPS, there is a great deal of freedom in $P_{1,i}$. This is because $P_{1,i}$ is a two-dimensional function with only one-dimensional constraints (e.g. Benson et al. 2005). We emphasize that this is true also for small time-steps. Hence there are many solutions for the desired EPS merger rates. In the next section we show several valid solutions of this sort.

3 SPECIFIC SOLUTIONS

Here we bring a general formalism for obtaining solutions $P_{1,i}$ and demonstrate the level of freedom allowed while obeying the EPS conditional mass functions. Given the robust expression in eq. (4) for P_{tot} in the limit of a small time-step, the solutions presented below are valid for any value of $\Delta\omega$ once it is small enough.

3.1 Determining a unique set of M_i 's for a given M_1

Our general solution is motivated by the merger-rate concept introduced by LC93. Assume that for any M_1 we can choose a *unique* set of smaller progenitors $\{M_i\}$, so that each $P_{i|1}$ is a delta function:

$$P_{i|1}(M_i|M_1, M_0, \Delta\omega) = \delta[M_i - f_i(M_1|M_0, \Delta\omega)] \quad (14)$$

Here $f_i(M_1|M_0, \Delta\omega)$ associates a value of M_i to any M_1 . We often write $f_i(M_1)$ where M_0 and $\Delta\omega$ are obvious from the context. Substituting $P_{i|1}$ from eq. (14) in the constraint of eq. (10), and integrating over M_1 , we obtain a differential equation for $f_i(M_1)$:

$$\frac{df_i(M_1)}{dM_1} = -\frac{P_1(M_1)}{P_i[f_i(M_1)]}, \quad (15)$$

where f_i is assumed to be a monotonically decreasing function of M_1 . Thus, the solution for $f_i(M_1)$ is determined by P_1 , P_i , and a certain initial condition $M_{i,0} = f_i(M_{1,0})$. This differential equation is to be integrated numerically to obtain a solution for $f_i(M_1)$. Note, in contrast, that LC93 adopted the inaccurate assumption $f_2^{LC}(M_1) = M_0 - M_1$, failing to allow for the additional progenitors beyond M_2 .

We start, for example, with $P_{2|1}$, using our default sharp-tail solution for P_1 as in eq. (9). Given this P_1 , we try to set $P_2 = P_{\text{tot}} - P_1$, which simply equals P_{tot} in the range $M < x_1 M_0$. A solution for $f_2(M_1)$ can now be obtained for a given initial condition. Our first choice, which we term ‘‘Solution I’’, is

$$(M_{1,0}, M_{2,0}) = (x_1 M_0, x_1 M_0). \quad (16)$$

This ensures that M_2 approaches M_1 as the latter obtains its minimum value $x_1 M_0$. Solution I is shown in

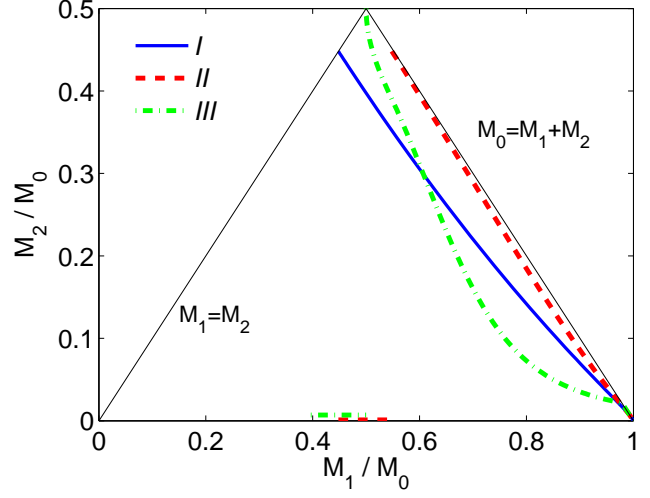


Figure 3. Three solutions for $P_{1,2}$, with a unique M_2 for each M_1 . The solutions are derived here for $M_0 = 10^{13} h^{-1} M_\odot$, $\Delta\omega = 10^{-6}$; they are practically the same for any smaller $\Delta\omega$. The solid (blue) and dashed (red) curves are computed for the same P_1 (the default sharp tail), and they differ only in the initial conditions (termed solutions I and II). The dotted-dashed (green) curve is obtained using the linear tail for P_1 (solution III). Note that the dashed and dotted-dashed lines have disconnected segments near $(M_1, M_2) \sim (M_0/2, 0)$. The solid line is our default solution (I). A summary of these solutions can be found in table 1.

Fig. 3. We also plot the solution for the initial condition $(M_{1,0}, M_{2,0}) = (M_0 - x_1 M_0, x_1 M_0)$, termed solution II. As is evident from the figure, although both solutions have the same P_1 , they have quite different values of $P_{1,2}$. Figure 3 also shows Solution III, which is based on P_1 with the ‘‘linear tail’’ shown in Fig. 2. A summary of these three solutions is listed in table 1. It should be noted that solutions II and III include a small range of M_1 values near $M_0/2$ that is not connected to M_2 through $M_2 = f_2(M_1)$. For M_1 in this range we cannot use any value of M_2 that was previously associated with M_1 through f_2 . We can either choose $M_2 \equiv 0$, or treat it similarly to P_3 , as will be explained below.

Luckily, the condition for mass conservation is almost fully obeyed by each of the three solutions for $f_2(M_1)$ above. This is implied by the fact that the curves for $P_{1,2}$ seem to always lie below the line $M_1 + M_2 = M_0$. However, a closer look shows that this constraint is violated for $M_1 \gtrsim 0.99 M_0$. In this range $f_2(M_1) > M_0 - M_1$ for all the solutions presented here, and we cannot adopt the M_2 that solves the differential equation. Instead, we enforce $M_2 = M_0 - M_1$, which makes the distribution P_2 differ slightly from P_{tot} . This result is expected based on the multiple-progenitor theorem of §2.2, requiring more than two progenitors for reproducing P_{tot} .

Fig. 4 shows P_{tot} and P_2 for small M values. The effect discussed above leads to $P_2 < P_{\text{tot}}$ at $M_2 < 0.01 M_0$, meaning that additional progenitors are needed in order to obtain an accurate fit for P_{tot} .

Next we should address P_i for the i -th progenitor, $i > 2$. In what follows we use as an example the solution I of $P_{1,2}$, and the procedure can be easily generalized to deal with the other solutions. For $i > 2$ we define

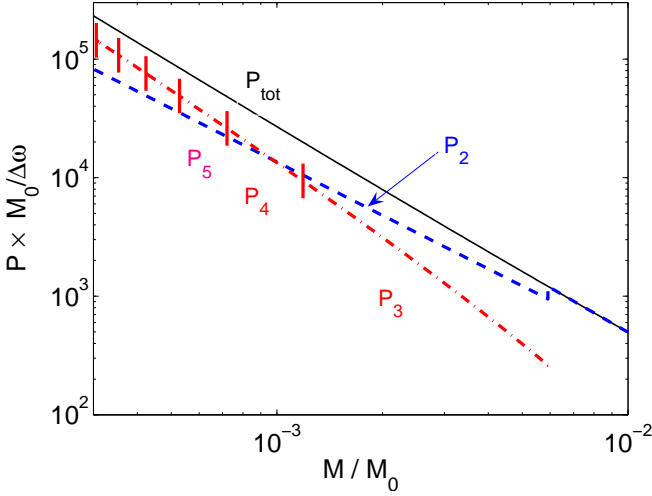


Figure 4. The probability distribution for small progenitors according to our algorithm, eqs. (17) and (18), using $M_0 = 10^{13} h^{-1} M_\odot$ and $\Delta\omega = 10^{-6}$. The solid line is P_{tot} . The dashed (blue) curve is P_2 . It deviates from P_{tot} for $M \lesssim 6 \times 10^{-3} \times M_0$. The Dashed-dotted curves correspond to P_i for $i > 2$. They equal $P_{\text{tot}} - P_2$. The vertical bars mark the limits for each P_i , termed $M_{\text{high},i}$ and $M_{\text{low},i}$.

Table 1. The characteristics of the EPS solutions for $P_{1,2}$ discussed in §3.1. The three solutions assume the delta-function form for $P_{2|1}$, eq. (14).

Solution	P_1 tail	$P_{1,2}$ Initial conditions
<i>I</i>	Sharp	$(x_1 M_0, x_1 M_0)$
<i>II</i>	Sharp	$(x_1 M_0, 1 - x_1 M_0)$
<i>III</i>	Linear	$(M_0/2, M_0/2)$

$$P_i(M) = \begin{cases} P_{\text{tot}}(M) - P_2(M) & M_{\text{low},i} \leq M < M_{\text{high},i} \\ 0 & \text{otherwise} \end{cases}, \quad (17)$$

where $M_{\text{high},i} = M_{\text{low},i-1}$ for $i > 3$ and $M_{\text{high},3}$ is the maximum M for which $P_2 < P_{\text{tot}}$. The value of $M_{\text{low},i}$ is set by the condition of mass conservation: each solution $f_i(M_1)$ is defined up to the point where $M_0 = M_1 + \sum f_i(M_1)$. The initial condition is thus

$$(M_{1,0}, M_{i,0}) = (x_1 M_0, M_{\text{high},i}), \quad (18)$$

and the set of P_i we obtain is given in fig. 4.

To summarize, our solution for the merger rate is

$$P_{1,i}(M_1, M_i | M_0, \Delta\omega) = P_1(M_1 | M_0, \Delta\omega) \delta[M_i - f_i(M_1 | M_0, \Delta\omega)]. \quad (19)$$

In the limit of a small time-step, f_i does not depend on $\Delta\omega$ and P_1 is given by eq. (4).

3.2 Comparison with N -body results

We now wish to compare the merger-rates from our EPS analysis to merger-rates that were extracted from N -body simulations. We remind the reader that when using N -body simulations with small time-steps, the merger-rates suffer from inconsistencies due to non-Markov features (ND08), so

any Markov model, as the one implied by EPS, will have deviations at small time-steps. A more fair comparison should be done against the model derived by ND08, which should be close to the optimum Markov fit to the simulations.

In Fig. 5 we show results of our solutions *I* and *II* against N -body simulations (the Millennium run, Springel et al. 2005). In order to test our solutions we use the merger-tree algorithm as described in section 6 below. Fig. 7 of ND08 indicates that the EPS merger rates found here do resemble relatively well the merger rates of the Markov model that fits the simulation in ND08. At bigger time-steps, we see that although the general contour shape is similar, the average mass of the second progenitor is slightly smaller in EPS than in the simulation. This is also evident in the results of Parkinson et al. (2008), who compared a different set of N -body merger trees with a binary-merger model for EPS trees a la LC93. On the other hand, merger trees constructed using the algorithm of Somerville & Kolatt (1999) have a significantly lower mass for M_2 , as pointed out in ND08. We find that the algorithm proposed by Kauffmann & White (1993) produces EPS trees that match the N -body trees at a level comparable to our EPS solution *I*, though it may not be as useful as our algorithm in generating a statistical sample of merger trees and in allowing analytic estimates.

When comparing Fig. 7 of ND08 to Fig. 5 here, it seems that the most important difference lies in the shape of the main-progenitor distribution. We recall that according to ND08, for trees extracted from N -body simulations, this distribution is log-normal in S . On the other hand, in EPS this distribution is given by eq. (9), which has quite a different shape (see also Fig. 2).

Figure 5 also compares solutions *I* and *II*, showing that solution *I* is somewhat closer to the N -body results. At the smaller time-step, solution *II* has slightly higher values of M_1 than the simulation, and it also has an isolated peak near $(M_1, M_2) = (M_0/2, 0)$, with no parallel trace in the simulation. At $\Delta\omega = 1.7$, solution *II* shows bigger deviations in the masses of M_1 and M_2 . Based on these findings, we adopt solution *I* as our default option for $P_{1,2}$. However, one should bear in mind that each of the three solutions discussed above is an example of a solution that is fully consistent with the EPS conditional mass functions.

3.3 A More Realistic Model?

The solution in terms of delta-functions, eq. (14), is motivated by the work of LC93 and by results from N -body simulations. We find that the $P_{1,2}$ extracted from the Millennium simulation indeed approaches a narrow function when $\Delta\omega \rightarrow 0$. Nonetheless, it should be noted that the delta-function solution is not the only possible solution for EPS even when $\Delta\omega \rightarrow 0$. We do find other EPS solutions with a broad $P_{2|1}$. The delta-function treatment is simple, though it has its limitations. For the finite time-steps used, the actual width of the distribution in the N -body simulation is finite, not zero. With the optimal time-step for reconstructing merger trees, $\Delta\omega \sim 0.1$ (ND08), the delta-function solution is accurate within EPS, but it is not such a good approximation to the N -body merger trees.

A different approach might be to seek a solution that can be used with any time-step $\Delta\omega$ in a self-consistent way, namely it should keep the same when using k time-steps of

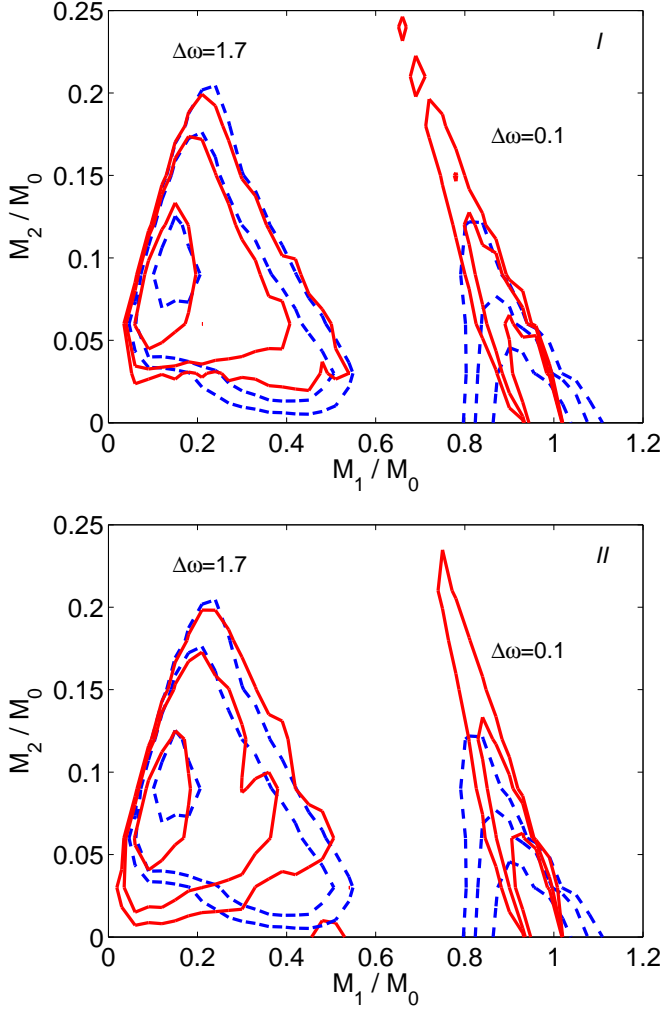


Figure 5. The joint probability of the two most massive progenitors, $P_{1,2}$, for haloes of mass $2 \times 10^{13} h^{-1} M_{\odot}$. The plots refer to two different time-steps, $\Delta\omega = 0.1$ and 1.7 . The contour levels are at $P_{1,2} = 5, 10, 30 M_0^{-2}$. *Upper panel:* The solid (red) contours refer to results of the EPS algorithm *I*, obtained from generated random realization with intrinsic $\Delta\omega = 0.001$. *Lower panel:* Same for solution *II*. The corresponding results from the Millennium *N*-body simulation are shown for comparison as dashed (blue) contours (previously presented in ND08, Fig. 7).

$\Delta\omega_1$ or one time-step of $\Delta\omega = k \times \Delta\omega_1$. Motivated by ND08, we try

$$P_{2|1}(M_2|M_1, M_0, \Delta\omega) = P_1(M_2|M_a, \Delta\omega), \quad (20)$$

where $M_a = f_2(M_1)$ as defined in eq. (14). This solution is fully consistent with P_{tot} for small enough $\Delta\omega$ as it approaches a delta function. However, for big time-steps it shows some deviations from the theoretical P_{tot} , depending on the specific solution adopted for $P_{1,2}$. This solution is not practical for our applications because it does not fit accurately the shape of $P_{1,2}$ as obtained from many small time-steps of solution *I*. We mention it here because it is close to solution *II* even for big time-steps.

For completeness, the explicit expression for the merger rate in this case is

$$P_{1,2}(M_1, M_2|M_0, \Delta\omega) = \frac{1}{2\pi} \frac{M_0 M_a}{M_1 M_2} \frac{(\Delta\omega)^2}{(\Delta S_1 \Delta S_2)^{1.5}} \quad (21)$$

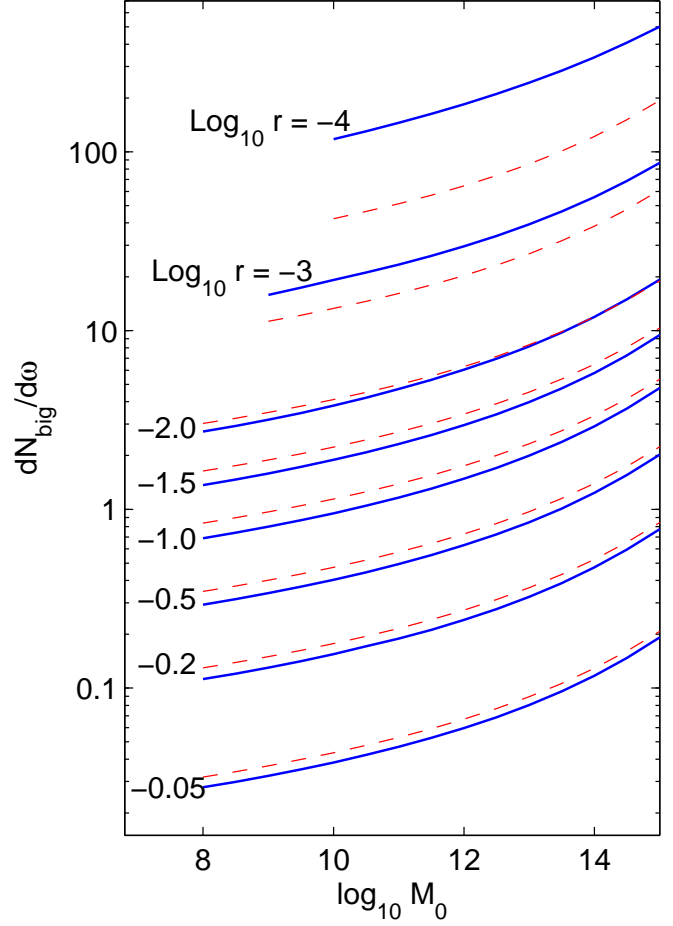


Figure 6. The number of merger events with mass ratio $> r$, per unit “time” $d\omega$, for a given final halo mass M_0 (in units of $h^{-1} M_{\odot}$). The value of r associated with each pair of curves is indicated. The solid curves describe the results of our EPS model. The dashed curves are the results of the LC93 formula, assuming that the main progenitor is more massive than $0.5M_0$ and $f_2^{LC} = M_0 - M_1$.

$$\times \exp \left[-\frac{(\Delta\omega)^2}{2} \left(\frac{1}{\Delta S_1} + \frac{1}{\Delta S_2} \right) \right] \left| \frac{dS(M_1)}{dM_1} \right| \left| \frac{dS(M_2)}{dM_2} \right|,$$

where $\Delta S_1 = S(M_1) - S(M_0)$ and $\Delta S_2 = S(M_2) - S(M_a)$.

4 PRACTICAL APPLICATIONS

We next implement the method outlined above for EPS merger rates, specifically solution *I*, to compute several quantities concerning the clustering of dark-matter haloes, which are of practical interest in the studies of galaxy formation.

4.1 Major and Minor Merger Rates for a Given Halo M_0

We first compute the probability that a halo of mass M_0 has undergone within the last time-step $\Delta\omega$ a merger event that includes the main-progenitor M_1 and another progenitor of

mass $M_i > rM_1$ ($i \geq 2$). Using our definition for merger rate, section 2.3, this can be written as

$$\frac{dN_{\text{big}}(r, M_0)}{d\omega} = \frac{d}{d\omega} \sum_{i=2}^{\infty} \int_{M_i > M_1 r} P_{1,i}(M_1, M_i | M_0, \Delta\omega) dM_1 dM_i. \quad (22)$$

We use eq. (19) in order to integrate over M_i and obtain

$$\frac{dN_{\text{big}}(r, M_0)}{d\omega} = \sum_i \int_{f_i(M_1) > M_1 r} \frac{dP_1(M_1 | M_0)}{d\omega} dM_1. \quad (23)$$

Eq. (5) provides the derivative of P_1 with respect to ω for any $r > 0$. This rate is independent of redshift as it is expressed in terms of the self-invariant time variable ω . When needed in units of time, one should multiply the above expression by $\dot{\omega}$. A useful approximation for $\dot{\omega}$ (from ND08) is

$$\dot{\omega} = -0.0470 [1 + z + 0.1(1 + z)^{-1.25}]^{2.5} h_{73} \text{ Gyr}^{-1}, \quad (24)$$

where h_{73} is the Hubble constant in units of 73 km s^{-1} . This approximation is valid for the ΛCDM cosmology used here, with $(\Omega_m, \Omega_\Lambda) = (0.25, 0.75)$, to better than 0.5% at all redshifts.

Figure 6 shows results for $dN_{\text{big}}(r, M_0)/d\omega$. For example, we read that $dN_{\text{big}}(0.3, 10^{12} h^{-1} M_\odot)/d\omega \sim 0.65$, which means that a halo of mass $10^{12} h^{-1} M_\odot$ has undergone on average 0.65 major mergers of $r > 0.3$ per unit of ω . Multiplying by $\dot{\omega}$ at $z = 0$ gives 0.04 major mergers per Gyr. At $z = 3$ it yields ~ 1 such mergers. The number of minor mergers, with $10^{-4} < r < 0.3$, is drastically higher; a $10^{12} h^{-1} M_\odot$ halo has ~ 10 such minor mergers per Gyr at $z = 0$, and ~ 250 such events per Gyr at $z = 3$.

Figure 6 also shows the number of merger events as derived from the formula of LC93, and assuming that the main progenitor is more massive than $M_0/2$. The LC93 approach is interpreted here as $f_2^{\text{LC}}(M_1) = M_0 - M_1$. The error due to their assumption is $\sim 20\%$ for major mergers, and it becomes as large as a factor of ~ 3 at $r \sim 10^{-4}$. We emphasize that this is true for our default solution *I*. It is possible that another EPS solution may be somewhat closer to the LC93 results, but the discrepancy of the LC93 estimates for minor mergers is likely to remain large.

4.2 Growth Rate of a Halo M_0 due to Major Mergers

As a second example we compute the average mass fraction added to a halo by merger events with mass ratio greater than r ,

$$\frac{dF_{\text{big}}(r, M_0)}{d\omega} = \frac{d}{d\omega} \sum_{i=2}^{\infty} \int_{M_i > M_1 r} P_{1,i}(M_1, M_i | M_0, \Delta\omega) \frac{M_i}{M_0} dM_1 dM_i. \quad (25)$$

As before, we can simplify the expression to

$$\frac{dF_{\text{big}}(r, M_0)}{d\omega} = \sum_i \int_{f_i(M_1) > M_1 r} \frac{dP_1(M_1 | M_0)}{d\omega} \frac{f_i(M_1)}{M_0} dM_1. \quad (26)$$

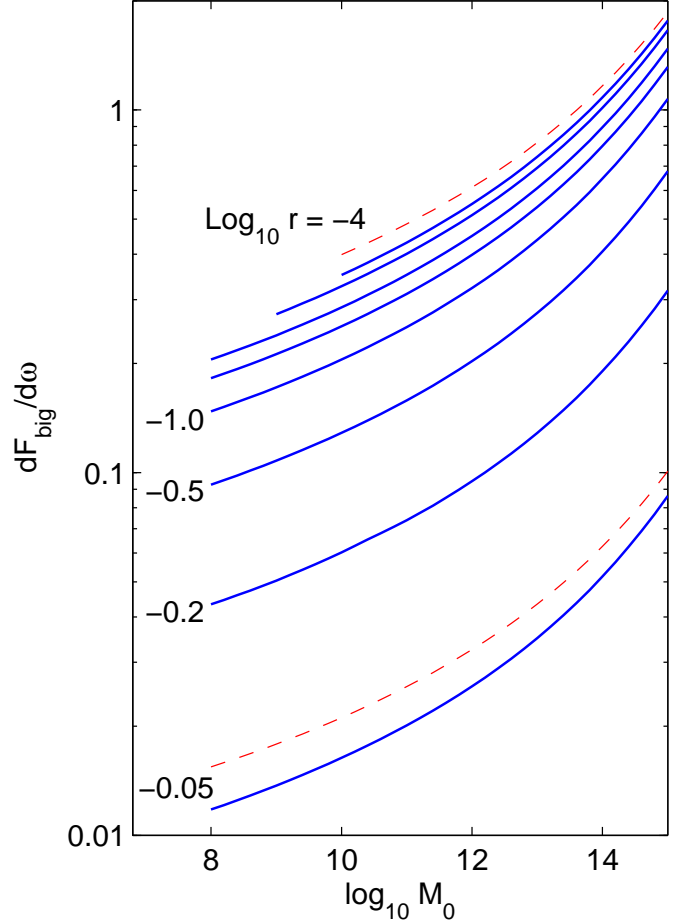


Figure 7. The mass fraction added to a halo of mass M_0 by mergers with progenitors of mass ratio $> r$. The values of r are the same as in Fig. 6. The solid curves describe our EPS model. The dashed curves refer to LC93; they are plotted only for the extreme values of r .

Results for $dF_{\text{big}}(r, M_0)/d\omega$ are shown in Fig. 7. As an example, $dF_{\text{big}}(0.3, 10^{12} h^{-1} M_\odot)/d\omega$ reads ~ 0.2 . This means that a halo of mass $10^{12} h^{-1} M_\odot$ has gained on average $\sim 20\%$ of its mass by major mergers per unit of ω . Multiplying by $\dot{\omega}$ we get a growth rate of 1% per Gyr by major mergers at $z = 0$, and $\sim 30\%$ at $z = 3$. For this quantity the LC93 assumption leads to similar errors of $\lesssim 20\%$ for all mass ratios r .

4.3 Merger Rates for a given M_1

The number of haloes of mass M_s that will merge with a halo of a given mass M_1 ($M_s < M_1$) within the time-step $\Delta\omega$, with no restriction on the descendant mass M_0 , is

$$\frac{dQ(M_s | M_1, z)}{d\omega} = \int_{M_1 + M_s}^{M_1/x_1} \frac{dP_{1,s}(M_1, M_s | M_0, \Delta\omega)}{d\omega} \frac{\phi(M_0, z)}{\phi(M_1, z)} dM_0, \quad (27)$$

where $\phi(M, z)$ is the Press-Schechter average comoving number density of haloes of mass M at redshift z , eq. (3). Using $P_{1,i}$ from eq. (19) we get

$$\frac{dQ(M_s|M_1, z)}{d\omega} = \sum_i \frac{dP_1(M_1|M_{0,i})}{d\omega} \frac{\phi(M_{0,i}, z)}{\phi(M_1, z)} \left| \frac{df_i}{dM_0} \right|^{-1}, \quad (28)$$

where $M_{0,i}$ are the values of M_0 for which $M_s = f_i(M_1|M_{0,i})$. Note that the derivative df_i/dM_0 is with respect to M_0 rather than M_1 .

Figure 8 shows results for $dQ(M_s|M_1, z)/d\omega$. Unlike the other quantities discussed above, Q does depend explicitly on redshift z , through the dependence of the sum in eq. (28) on z . Nevertheless, for major mergers (high r) this sum consists of only one term, so the z dependence can be scaled out. Note also that Q is not a smooth function, due to the fact that $f_i(M_1)$ are always defined for an M_1 value that is smaller than some threshold, in order to conserve mass (see the discussion after eq. (18)). Figure 8 displays in comparison the results of the LC93 formula, showing deviations of $\sim 30\%$ for major mergers, which become as large as a factor of ~ 3 for a small mass ratio of $r \sim 10^{-4}$.

5 CREATION AND DESTRUCTION RATES OF HALOES

In this section, we address the merger processes through the creation and destruction rates of haloes. We show that the results obtained here are consistent with the Press-Schechter mass function, as they should be. This can also serve as a sanity check for validating the values of Q computed above.

Benson et al. (2005) have attempted to use the Smoluchowski coagulation equation for computing halo merger rates. This equation evaluates the change in the number density of haloes due to the competing processes of halo creation and destruction as a result of mergers. Their formulae assume that each halo is formed by a binary merger event, so halo formation is modeled as a 2-progenitor process. According to the multiple progenitor theorem proved above, this approach cannot yield correct results. A formulation of the Smoluchowski coagulation equation should be therefore replaced with a different equation that takes into account multiple mergers and accretion mass. Consequently, the discrepancy found by Benson et al. (2005) in the merger rate formula of LC93 is not a discrepancy in the EPS formalism – it simply reflects the inaccuracy introduced in the LC93 formula by the assumption of binary mergers. Despite this built-in error, the numerical merger rates by Benson et al. (2005) seem to be consistent with the Press-Schechter mass function. This could be the result of using a mass grid cell size of $M_0/179$, which scales with mass, while we found that multiple mergers become relevant only for $M < 10^{-3}M_0$.

The EPS formalism is, by construction, fully consistent with the Press-Schechter mass function. This can be expressed by

$$\phi(M, \omega + \Delta\omega) = \int_M^\infty \phi(M_0, \omega) P_{\text{tot}}(M|M_0, \Delta\omega) dM_0. \quad (29)$$

This equation indicates that any merger rate that is consistent with P_{tot} must be consistent with the way ϕ varies in

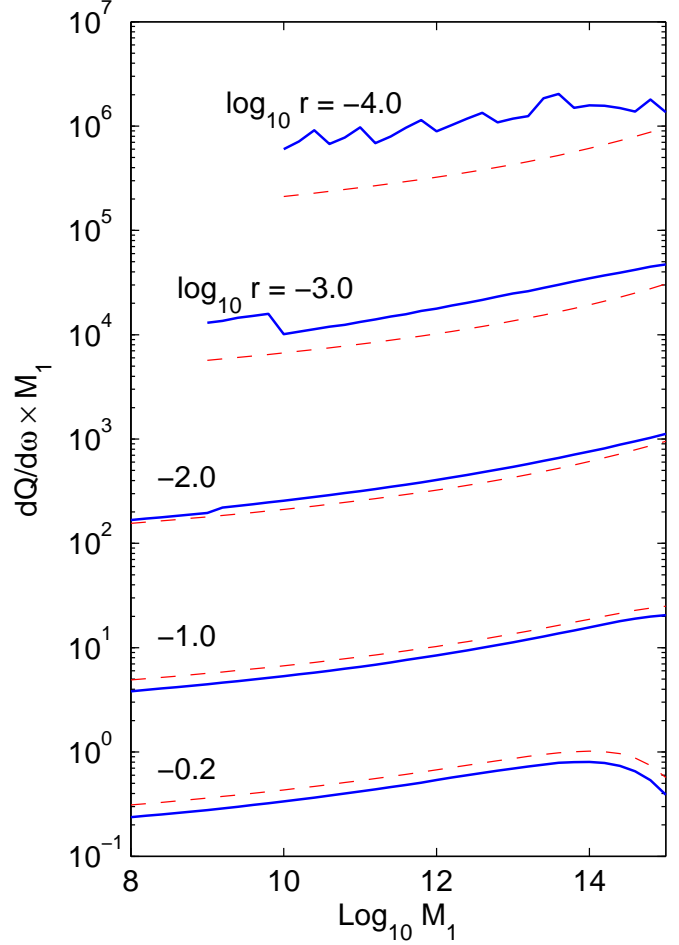


Figure 8. The number of mergers of mass $M_s = rM_1$ with a given halo of mass M_1 , in an infinitesimal time-step $d\omega$. The results are plotted for $z = 0$. The solid curves are results of the EPS solution I given in §3. The dashed curves follow the formula of LC93.

time. This implies that the merger rates that were evaluated here should predict the correct rate of change of ϕ when implemented using halo creation and destruction terms.

Taking into account the multiple progenitors and the accreted mass, the time derivative of $\phi(M, z)$ is connected to Q via the equation

$$\begin{aligned} -\frac{d\phi(M, z)}{d\omega} = & \lim_{\Delta M \rightarrow 0, \Delta\omega \rightarrow 0} \frac{1}{\Delta\omega} \frac{1}{2\Delta M} \left[\int_{M-\Delta M}^{M+\Delta M} \phi(M_0, z) dM_0 \int_0^{M-\Delta M} P_1(M_1|M_0, \Delta\omega) dM_1 \right. \\ & - \int_{M+\Delta M}^\infty \phi(M_0, z) dM_0 \int_{M-\Delta M}^{M+\Delta M} P_1(M_1|M_0, \Delta\omega) dM_1 \left. \right] \\ & - \int_M^\infty \frac{dQ(M|M_1, z)}{d\omega} \phi(M_1, z) dM_1. \end{aligned} \quad (30)$$

The first term corresponds to the creation of new haloes inside the mass bin $[M - \Delta M, M + \Delta M]$ as arising from the main-progenitor growth rate. The second term computes

the number of haloes that leave this bin for the same reason. We note that each of these terms diverges for small ΔM , but their sum remains constant. The third term that involves Q is the number of haloes that leave the mass bin by merging with bigger haloes. We have verified that this formula yields self-consistent results by computing it term by term. However, due to the numerical limitations of computing Q in only discrete points, we get an accuracy that is on the order of few percents in the integral of Q .

6 A MONTE-CARLO ALGORITHM FOR EPS MERGER TREES

Using the specific analytical solution obtained in §3, one can construct full merger trees. The algorithm is conceptually simple, and can be summarized as follows:

- Define a reference halo with mass M_0 at ω_0 .
- Choose a time-step $\Delta\omega$ (not necessarily small).
- Draw a random main-progenitor mass M_1 , using the distribution $P_1(M_1|M_0, \Delta\omega)$.
- Compute the value of M_i ($i \geq 2$) using $M_i = f_i(M_1|M_0, \Delta\omega)$, for every value of i , until the desired mass resolution is achieved.
- Repeat the above procedure for each progenitor M_i , where M_0 is replaced by M_i .

This general algorithm can be used with any variant of the solutions presented in §3. An advantage of this algorithm is that all the tree quantities can be computed analytically. Another advantage over other algorithms is that its accuracy within EPS is in principle unlimited — it solely depends on the accuracy of the f_i used. The algorithm can be applied with time-steps that are not small, but the procedure is simpler when using small time-steps so P_{tot} is linear in $\Delta\omega$.

Figure 9 shows results from EPS merger-tree realizations using our algorithm based on solution *I*. These results demonstrate the high accuracy of the generated trees.

7 SUMMARY AND DISCUSSION

We presented a rigorous method for computing dark-matter merger rates and merger trees that obey the halo progenitor mass function of the EPS formalism at any redshift. This corrects apparent inconsistencies within EPS (Lacey & Cole 1993; Benson et al. 2005). Our method conserves mass, in the sense that the sum of the progenitor masses does not exceed the mass of the product halo. This method translates the problem of constructing merger trees to solving a differential equation. Different choices of initial conditions correspond to different types of merger trees. This method enabled us to span the set of solutions for merger rates within EPS, and to pick up a specific solution whose merger trees are a good fit to N -body results. The same method can be implemented with any conditional mass function beyond EPS, e.g., as extracted from N -body simulations or from an ellipsoidal-collapse model.

Our main result is an accurate derivation for the merger rate of dark-matter haloes, which differs from the classical result of Lacey & Cole (1993). This is due to our finding that within the EPS formalism, a merger event typically involves

many progenitors in a time-step, even when this time-step is infinitely small, as opposed to the binary mergers assumed in previous works. Our corrected results differ from those derived by Lacey & Cole (1993) especially in the number of minor merger events, while other quantities deviate only at the level of 20%. We compute a few useful variants of the merger-rate formula, such as the number of mergers for a given descendant halo, the mass fraction added by mergers, and the merger rate per progenitor halo. These examples span many applications for galaxy formation models. We also verified that the merger rates derived here are fully consistent with the evolution of the Press-Schechter mass function, in terms of counting the creation and destruction of haloes within the coagulation equation.

We have shown that the merger rates derived here fit the results of N -body simulations better than the early results of Lacey & Cole (1993). However, as discussed in Neistein & Dekel (2008), the merger rates from N -body simulations may suffer from intrinsic inconsistencies at the level of a few tens of percents due to non-Markov effects. Keeping this in mind, it is tempting to compare our EPS merger rates with other studies of merger-rates extracted from N -body simulations (e.g. Fakhouri & Ma 2007; Stewart et al. 2007). For example, it is likely that our EPS results are in better agreement with the N -body results than the EPS results presented by Fakhouri & Ma (2007); their EPS merger rates are underestimates at low mass ratio of $r \sim 10^{-3}$. This better agreement is similar to what we find here based on the merger-rates of Neistein & Dekel (2008).

The concept of multiple mergers in the limit of small time-steps, proven here to be valid in EPS, deserves further attention. Recent studies indicate that this might be true in N -body simulations when the time-steps used are finite (e.g. Fakhouri & Ma 2007; Neistein & Dekel 2008). However, in an N -body system, every merger event can be broken into a sequence of binary mergers once the time step is short enough. This implies that the N -body system is not a pure Markov process — the binary mergers are a non-Markov feature. This non-Markov feature reflects a correlation between the successive mergers. A correlation of this sort may be introduced, for example, by the progenitors being part of a cosmic-web filament feeding a bigger halo, where they merge in as a coherent group. When we impose a Markov model to describe the N -body mergers, i.e. a model that ignores any correlations, this correlated sequence of binary mergers is forced to appear as a multiple merger. It would be interesting to verify this interpretation of the relation between the EPS and N -body mergers by testing whether most of the N -body merger events are indeed part of a correlated sequence.

If multiple-merger events are a common phenomena in EPS, then the merger rates are not defined in a unique way, as the counting method by which progenitors are ordered to merge with each other may affect the merger-rate results. Here we have chosen a simplified approach, where all the progenitors are assumed to merge with the most massive progenitor and not with one another. Clearly, other methods of counting may be applied. This issue may be examined in detail using an N -body simulation, where the multiple mergers can be broken into a sequence of binary events once the time steps are made small enough. It should be noted that the conditional mass function of progenitors as extracted from N -

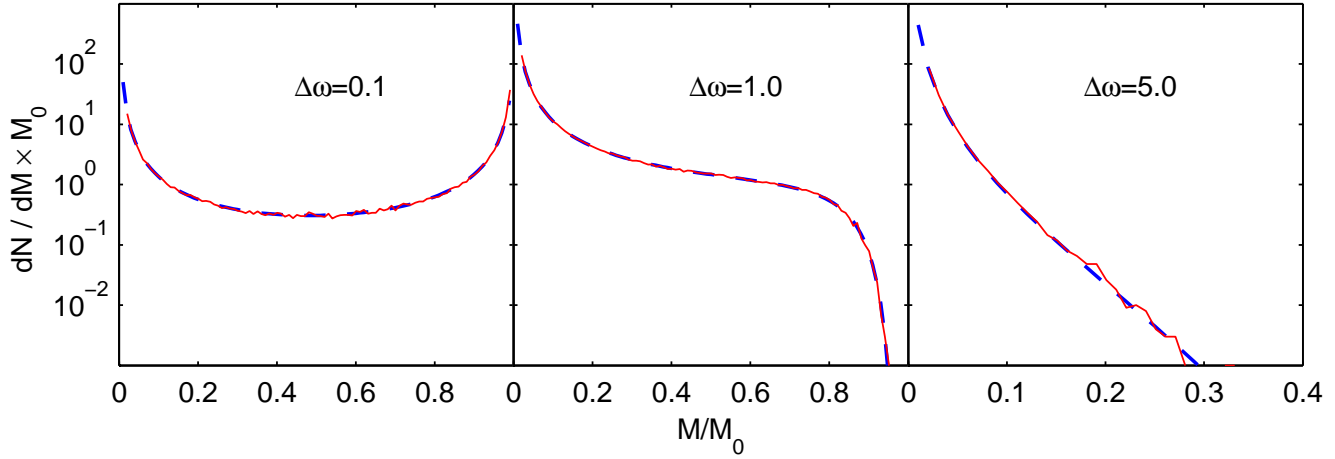


Figure 9. The number of progenitors, eq. (1), using $M_0 = 10^{13} h^{-1} M_\odot$ and three different values of $\Delta\omega$. The results based on 10^5 random realizations with intrinsic time-step $\Delta\omega = 0.005$ are shown as solid (red) thin lines. The dashed line is the theoretical prediction eq. (1).

body simulations cannot be reconstructed by a Markov process (see Neistein & Dekel 2008). This means that there is no accurate expression for this mass function at small time-steps that can reproduce the mass function at high redshift, namely separated from the present by a large time-step (but see Cole et al. 2008, for an approximation). Further effort is needed in order to understand this issue in N -body simulations.

A conditional mass function that is based on the ellipsoidal collapse model has been used recently for generating merger trees (Moreno & Sheth 2007) and for computing merger rates (Zhang et al. 2008). The use of the ellipsoidal model is partly motivated by its earlier success, over the spherical model used by Press-Schechter, in reproducing the (un-conditional) mass function of haloes in N -body simulations (Sheth & Tormen 2002). The method developed in this paper can be easily generalized to utilize the ellipsoidal collapse model. The results should be compared to our EPS predictions and to the N -body results. As a first step, it should be interesting to evaluate the level of accuracy in previous studies due to the binary-merger assumption by computing the average number of progenitors per merger event.

The algorithm we provide for generating merger trees has several advantages as follows:

- It is fully consistent with the EPS conditional mass function of progenitors.
- The relevant statistics can be described analytically, including those concerning the main-progenitor history and the merger rates.
- This algorithm was chosen, out of the many options that are consistent with EPS, to provide best fit to N -body simulations.
- The constructed merger trees conserve mass, in the sense that the total mass in progenitors does not exceed the descendant halo mass.

These are significant improvements over previous algorithms that follow EPS (Cole 1991; Kauffmann & White

1993; Sheth & Lemson 1999; Somerville & Kolatt 1999; Cole et al. 2000; Hiotelis & Popolo 2006), which makes the new algorithm a useful tool for analytic and semi-analytic modeling of galaxy formation. Still, the non-EPS algorithms that are empirically tuned to match N -body simulations (Parkinson et al. 2008; Neistein & Dekel 2008) may have advantages in certain cases where the accuracy is important.

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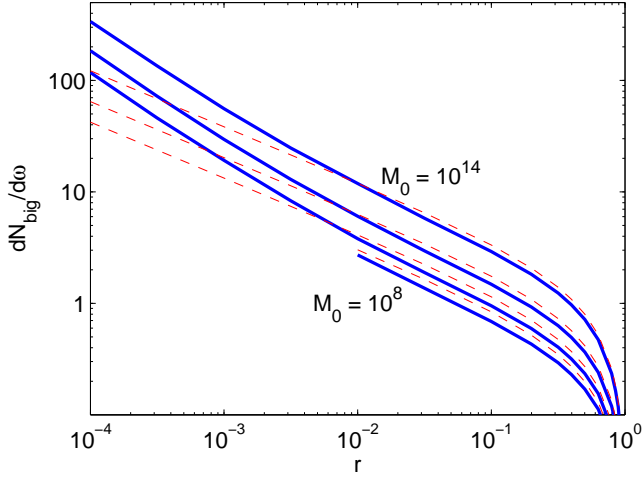


Figure A1. The number of merger events with mass ratio $> r$, per unit “time” $d\omega$, for a given final halo mass M_0 (in units of $h^{-1}M_\odot$). The values of M_0 are $10^8, 10^{10}, 10^{12}, 10^{14} h^{-1}M_\odot$. The solid curves describe the results of our EPS model while the dashed curves are the results of the LC93 formula.

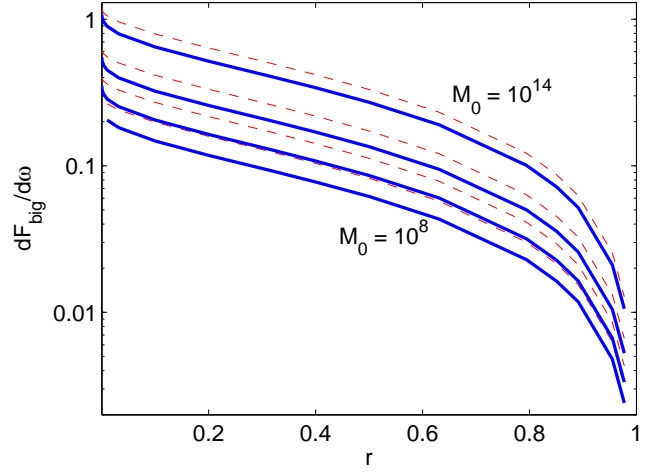


Figure A2. The mass fraction added to a halo of mass M_0 by mergers with progenitors of mass ratio $> r$. The values of M_0 are the same as in Fig. A1. The solid curves describe our EPS model and the dashed curves refer to LC93.

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APPENDIX A: MORE RESULTS

This Appendix is a supplement to §4, for the benefit of practitioners who desire to read out numerical values for merger rates from the figures. Figures 6-8 of §4 present merger-rate quantities as a function of halo mass for different given values of mass ratio r . Here we plot the same merger rate quantities as a function of r for different values of mass. This way of presenting the merger rates emphasizes their simple scaling with halo mass and highlights the trends at small values of r .

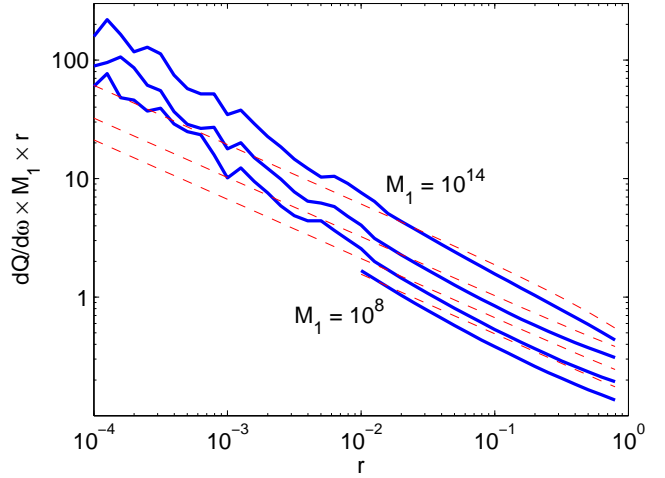


Figure A3. The number of mergers of mass $M_s = rM_1$ with a given halo of mass M_1 , in an infinitesimal time-step $d\omega$. The results are plotted for $z = 0$, and for $M_1 = 10^8, 10^{10}, 10^{12}, 10^{14} h^{-1}M_\odot$. The solid curves follow the EPS solution I given in §3, and the dashed curves are obtained from the formula of LC93.